Dynamic Modelling of Planar Mechanisms Using Point Coordinates

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In the present study, the dynamic modelling of planar mechanisms that consist of a system of rigid bodies is carried out using point coordiantes. The system of rigid bodies is replaced by a dynamically equivalent constrained system of particles. Then for the resulting equivalent system of particles, the concepts of linear and angular momentums are used to generate the equations of motion without either introducing any rotational coordinates or distributing the external forces and force couples over the particles. For the open loop case, the equations of motion are generated recursively along the open chains. For the closed loop case, the system is transformed to open loops by cutting suitable kinematic joints with the addition of cut-joints kinematic constraints. An example of a multi-branch closed-loop system is chosen to demonstrate the generality and simplicity of the proposed method.

Key Words : Dynamic Analysis, Recursive Formulation, Equations of Motion, System of Rigid Bodies, Open-Chain, Closed-Chains

1. Introduction

Many formulations have been used to carry out the dynamic analysis of planar mechanisms. Some formulations (Dix and Lehman, 1972 ; Orlandea et al., 1977 ; Nikravesh, 1988) use a large set of dependent coordinates. The location of each rigid body in the system is described in terms of a set of absolute coordinates ; translational and rotational coordinates. The constraint equations are imposed to represent the kinematic joints that connect the rigid bodies. This formulation has the advantage that the constraint equations are easily introduced, however, it has the disadvantage of a large number of coordinates defined. Other formulations (Denavit and Hartenberg, 1955 ; Sheth and Uicker, 1972) describe the configuration of the system in terms of relative coordinates. The location of each body is defined with respect to the adjacent body by means of an angle or a distance depending on the type of the kinematic pair joining the two bodies. Although this formulation yields the constraints as a minimal set of algebraic equations, it has the disadvantage that it does not directly determine the positions of the bodies and points of interest.

Other methods for generating the equations of motion use a two-step transformation. They group the advantages of the simplicity, generality and efficiency. One method (Jerkovsky, 1976 ; Kim and Vanderploeg, 1986 ; Nikravesh and Gim, 1989) uses initially the absolute coordinate formulation. Then, the equations of motion are expressed in terms of the relative joint variables. Another method (Attia, 1993 ; Nikravesh and Attia, 1994) uses initially a dynamically equivalent constrained system of particles to replace the rigid bodies. The mass associated with each particle is determined as a function of the inertia characteristics of each body. The external forces

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and couples acting on the body are transformed to equivalent forces and redistributed over the system of particles. The equations of motion are derived using Newton's second law and the Lagrange multiplier technique which results in a large number of differential-algebraic equations. The simplicity and the absence of any rotational coordinates from the final form of the equations of motion are considered the main advantages of this formulation. For the purpose of computational efficiency, the equations of motion that are expressed in matrix form in terms of the Cartesian coordinates of the particles are rederived in terms of the relative joint variables. The main disadvantage of these two-step transformations is the necessity to transform at every time step from the joint variables to the original system (the absolute coordinates or the Cartesian coordinates of the particles). This transformation process which is known as the forward process (Nikravesh and Gim, 1989 ; Attia, 1993) is very time consuming.

In the present paper, the dynamic modelling of planar mechanisms that consist of a system of rigid bodies with all common types of kinematic joints such as revolute and prismatic is carried out using point coordinates. The method rests upon the idea of replacing the rigid body by its dynamically equivalent constrained system of particles discussed in (Attia, 1993 ; Nikravesh and Attia, 1994). However, instead of the matrix form of the equations of motion derived in (Attia, 1993 ; Nikravesh and Attia, 1994), a more efficient recursive approach is used. For the resulting equivalent constrained system of particles, the concepts of linear and angular momentums are used to generate the equations of motion without either introducing any rotational coordinates or distributing the external forces and couples over the particles. For the open loop case, the equations of motion are generated recursively along the open chains instead of the matrix formulation derived in (Attia, 1993). Geometric constraints that fix the distances between the particles are introduced while some kinematic constraints due to commom types of joints and the associated constraint forces are automatically eliminated by properly selecting the locations of the particles. For the closed loop case, the system is transformed to open loops by cutting suitable kinematic joints and introducing the cut-joint kinematic constraints. The special case of a system of rigid rods is also discussed. The dynamic analysis of a multi-branch closed-loop mechanism is carried out to demonstrate the generality and simplicity of the suggested method.

2. The Dynamic Model

2.1 Construction of the equivalent system of particles

The requirements of equivalence of the mass, the position of center of mass, and the elements of the inertia tensor in the plane motion lead in general to six conditions. In this case, as shown in Fig. 1, six particles are chosen to represent the

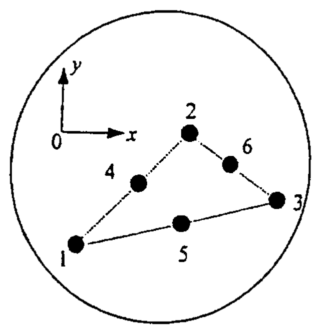

Fig. 1 The rigid body with its equivalent system of three particles

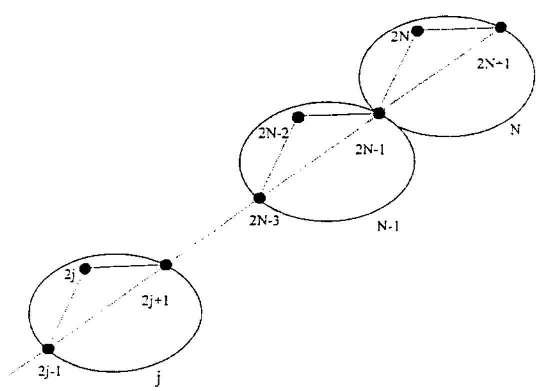

Fig. 2 Serial chain of N rigid bodies with the equivalent particales

rigid body in order to obtain a linear system of six equations in six unknown masses. Then the coordinates of the six particles become free variables and can be arbitrarily chosen to achieve additional requirements. Three particles 1, 2 and 3, named as primary particles, are located arbitrarily, while the others 4, 5 and 6, named as secondary particles, are conveniently located at the mid-points of the lines joining each pair of particles. By this arrangement, the six conditions are written as;

$$m=\sum_{i=1}^{6} m_i \tag{1a}$$

$$m\mathbf{r}_G=\sum_{i=1}^{6} m_i\mathbf{r}_i \tag{1b}$$

$$j_{\xi\xi}=\sum_{i=1}^{6} m_i\eta_i^2 \tag{1c}$$

$$j_{\eta\eta}=\sum_{i=1}^{6} m_i\xi_i^2 \tag{1d}$$

$$j_{\xi\eta}=\sum_{i=1}^{6} m_i\xi_i\eta_i \tag{1e}$$

where m is the mass of the rigid body, $\mathbf{r}_G$ is the position of the center of mass of the rigid body, $j_{\xi\xi}$, $j_{\eta\eta}$, $j_{\xi\eta}$ are the components of the inertia tensor of the rigid body with respect to the body attached coordinate frame, m_i is the mass of particle i and $\mathbf{r}_i$ is the position vector of particle i with respect to the body attached coordinate frame. The 6×6 linear system given by Eq. (1) can be put in the following matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \\ \eta_1^2 & \eta_2^2 & \eta_3^2 & \eta_4^2 & \eta_5^2 & \eta_6^2 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 & \xi_5^2 & \xi_6^2 \\ \xi_1\eta_1 & \xi_2\eta_2 & \xi_3\eta_3 & \xi_4\eta_4 & \xi_5\eta_5 & \xi_6\eta_6 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix} = \begin{bmatrix} m \\ m\xi_G \\ m\eta_G \\ j_{\xi\xi} \\ \xi_{\eta\eta} \\ j_{\xi\eta} \end{bmatrix} \tag{2}$$

It should be noted that in the case of a rigid rod, only three particles can dynamically replace it. In this case two particles can be located at both ends where the third particle can be located at the middle of the rod. Three linear algebraic equations can be solved for the three unknown masses of these particles.

2.2 Equations of motion of a single rigid body in plane motion

The rigid body, shown in Fig. 1, is replaced by an equivalent constrained system of six particles. The distances between the six particles are invariant as a result of the internal constraint forces existing between them. The vector sum of these unknown internal forces and also the vector sum of their moments about any point equals zero by the law of action and reaction (Goldstein, 1950). Then, the linear momentum equation for the whole system of particles yields,

$$\mathbf{R}=\sum_{i=1}^{6} m_i\ddot{\mathbf{r}}_i \tag{3}$$

where $\mathbf{R}$ is the vector sum of the external forces acting on the rigid body and $\ddot{\mathbf{r}}_i$ is the acceleration vector of particle i with respect to the global coordinate frame. Also, the angular momentum equation for the whole system of particles with respect to particle 1 takes the form (Goldstein, 1950)

$$\mathbf{G}_1=\sum_{i=2}^{6} \mathbf{r}_{i,1} x m_i\ddot{\mathbf{r}}_i \tag{4}$$

where $\mathbf{G}_1$ is directed perpendicular to the plane of motion and represents the vector sum of the moments of the external forces and force couples acting on the body with respect to the location of particle 1 and $\mathbf{r}_{i,1}$ is the relative position vector between particles i and 1.

The distance constraints between the six particles are given as

$$\mathbf{r}_{2,1}^T\mathbf{r}_{2,1}-d_{2,1}^2=0 \tag{5a}$$

$$\mathbf{r}_{3,1}^T\mathbf{r}_{3,1}-d_{3,1}^2=0 \tag{5b}$$

$$\mathbf{r}_{3,1}^T\mathbf{r}_{3,2}-d_{3,2}^2=0 \tag{5c}$$

$$\mathbf{r}_4-(\mathbf{r}_1+\mathbf{r}_2)/2=0 \tag{5d}$$

$$\mathbf{r}_5-(\mathbf{r}_1+\mathbf{r}_3)/2=0 \tag{5e}$$

$$\mathbf{r}_6-(\mathbf{r}_2+\mathbf{r}_3)/2=0 \tag{5f}$$

Differentiating Eq. (5) with respect to time leads to the velocity constraints

$$\mathbf{r}_{2,1}^T\dot{\mathbf{r}}_{2,1}=0 \tag{6a}$$

$$\mathbf{r}_{3,1}^T\dot{\mathbf{r}}_{3,1}=0 \tag{6b}$$

$$\mathbf{r}_{3,2}^T\dot{\mathbf{r}}_{3,2}=0 \tag{6c}$$

$$\dot{\mathbf{r}}_4-(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2)/2=0 \tag{6d}$$

$$\dot{\mathbf{r}}_5-(\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_3)/2=0 \tag{6e}$$

$$\dot{\mathbf{r}}_6-(\dot{\mathbf{r}}_2+\dot{\mathbf{r}}_3)/2=0 \tag{6f}$$

Differentiating Eq. (6) with respect to time leads to the acceleration constraints

$$\mathbf{r}_{2,1}^T(\ddot{\mathbf{r}}_2-\ddot{\mathbf{r}}_1)=-\dot{\mathbf{r}}_{2,1}^T\dot{\mathbf{r}}_{2,1} \tag{7a}$$

$$\mathbf{r}_{3,1}^T(\ddot{\mathbf{r}}_3-\ddot{\mathbf{r}}_1)=-\dot{\mathbf{r}}_{3,1}^T\dot{\mathbf{r}}_{3,1} \tag{7b}$$

$$\mathbf{r}_{3,2}^T(\ddot{\mathbf{r}}_3-\ddot{\mathbf{r}}_2)=-\dot{\mathbf{r}}_{3,2}^T\dot{\mathbf{r}}_{3,2} \tag{7c}$$

$$\ddot{\mathbf{r}}_4-(\ddot{\mathbf{r}}_1+\ddot{\mathbf{r}}_2)/2=0 \tag{7d}$$

$$\ddot{\mathbf{r}}_5-(\ddot{\mathbf{r}}_1+\ddot{\mathbf{r}}_3)/2=0 \tag{7e}$$

$$\ddot{\mathbf{r}}_6-(\ddot{\mathbf{r}}_2+\ddot{\mathbf{r}}_3)/2=0 \tag{7f}$$

The equations of motion (3), (4) and (7) represent a linear system of 12 scalar algebraic equations that can be solved to determine the unknown acceleration vectors $\ddot{\mathbf{r}}_i$, $i=1, \cdots, 6$ of the particles at any instant of time. They are expressed in terms of the rectangular Cartesian coordinates of the equivalent system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces as in Newton-Euler formulation while expresses the general motion of the rigid body in terms of a set of Cartesian coordinates without introducing any rotational coordinates. This process results in a reduced system of differential-algebraic equations and also eliminates the necessity of distributing the external forces and moments over the particles.

A reduced form of the equations of motion can be obtained by eliminating the secondary particles and their unknown accelerations through the substitution of the constraint Eqs. (7d) to (7f) into Eqs. (3) and (4) to obtain

$$\mathbf{R}=\sum_{i=1}^{3}\bar{m}_i\ddot{\mathbf{r}}_i \tag{8}$$

$$\mathbf{G}_1=\sum_{i=1}^{3}\mathbf{A}_i\ddot{\mathbf{r}}_i \tag{9}$$

where

$$\bar{m}_i=m_i+\sum_{\substack{j=1\\j\neq i}}^{3}\frac{1}{2}m_{j,i} \tag{10a}$$

$$\mathbf{A}_i=\bar{m}_i\tilde{\mathbf{r}}_{i,1}+\sum_{\substack{j=2\\j\neq i}}^{3}\frac{1}{4}m_{j,i}\tilde{\mathbf{r}}_{j,1} \tag{10b}$$

$$\bar{m}_i=m_i+\sum_{\substack{j=1\\j\neq i}}^{3}\frac{1}{4}m_{j,i} \tag{10c}$$

and where $m_{j,i}$ denotes the mass of the secondary particle that is located between the primary particles i and j ($m_{1,2}=m_{2,1}=m_5$, $\cdots$ etc.). Then, Eqs. (8) and (9) in addition to the remaining constraints Eqs. (7a) to (7c) represent the equations of motion for a single rigid body where only the primary particles stay.

The above linear system of Eqs. (8) and (9) represents the equations of motion of a single floating rigid body in plane motion. It can be solved at every time step to determine the unknown acceleration components of the remaining particles 1, 2 and 3. The acceleration components of the particles can be integrated numerically knowing their Cartesian coordinates and velocities at a certain time to determine the positions and velocities at the next time step (Hairer and Wanner, 2001). The rectilinear motion of the particles determines completely the translational and rotational motion of the rigid body. If the rigid body is rotating about a fixed axis, then particle 1 may be located at the intersection of the body with the axis of rotation. In this case, Eq. (9) is used to solve for the unknown Cartesian accelerations of particle 3. Then Eq. (8) can be solved to determine the unknown reaction forces at the axis of rotation. Also, for the case of a rigid rod the second intermediate particle can be eliminated and the equations of motion constitute of one moment equation, two scalar force equations, and one distance constraint. These four scalar equations are sufficient to solve for the unknown accelerations of the remaining end particles.

3. Equations of Motion of a Serial Chain of Rigid Bodies

Figure 3 shows a serial chain of N rigid bodies with the equivalent system of (2N+1) particles where connecting particles are unified from both bodies. For the last body "N" in the chain, the equations of motion are derived in a similar way

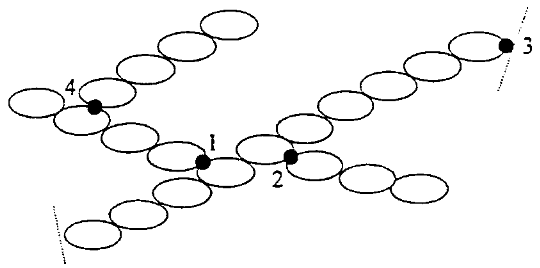

Fig. 3 A multi-branch system indicating four cut-joints

as Eq. (9) of a single rigid body. The angular momentum equation takes the form

$$\mathbf{G}_{2N-1}=\sum_{i=2N-1}^{2N+1}\mathbf{A}_i\ddot{\mathbf{r}}_i \qquad (11)$$

where

$$\mathbf{A}_h=\bar{m}_h\tilde{\mathbf{r}}_{h,2j-1}+\sum_{\substack{k=2i\\k\neq h}}^{2i+1}\frac{1}{4}m_{k,h}\tilde{\mathbf{r}}_{k,2j-1} \qquad (12a)$$

$$\bar{m}_h=m_h+\sum_{\substack{k=2i-1\\k\neq h}}^{2i+1}\frac{1}{4}m_{k,h} \qquad (12b)$$

where $\mathbf{G}_{2N-1}$ is the vector sum of the moments of the external forces and force couples acting on body N with respect to the location of particle $2N-1$. The distance constraints are given as

$$\mathbf{r}_{2N,2N-1}^T\mathbf{r}_{2N,2N-1}-d_{2N,2N-1}^2=0 \qquad (13a)$$

$$\mathbf{r}_{2N+1,2N-1}^T\mathbf{r}_{2N+1,2N-1}-d_{2N+1,2N-1}^2=0 \qquad (13b)$$

$$\mathbf{r}_{2N+1,2N}^T\mathbf{r}_{2N+1,2N}-d_{2N+1,2N}^2=0 \qquad (13c)$$

Addition of one more body in the chain leads to the inclusion of an angular momentum equation that takes into consideration the contributions of all the ascending bodies in the chain together with one distance constraint between the particles belonging to this body. These two equations are appended to the equations of motion derived for the leading bodies in the chain. For body j, the appended equations of motion take the form

$$\mathbf{G}_{2j-1}=\sum_{i=j}^{N}\sum_{k=2i-11}^{2i+1}\mathbf{A}_k\ddot{\mathbf{r}}_k \qquad (14)$$

where

$$\mathbf{A}_h=\bar{m}_h\tilde{\mathbf{r}}_{h,2N-1}+\sum_{\substack{k=2N\\k\neq h}}^{2N+1}\frac{1}{4}m_{k,h}\tilde{\mathbf{r}}_{k,2N-1} \qquad (15a)$$

$$\bar{m}_h=m_h+\sum_{\substack{k=2N-1\\k\neq h}}^{2N+1}\frac{1}{4}m_{k,h} \qquad (15b)$$

$$\mathbf{r}_{2j,2j-1}^T(\ddot{\mathbf{r}}_{2j}-\ddot{\mathbf{r}}_{2j-1})=-\dot{\mathbf{r}}_{2j,2j-1}^T\dot{\mathbf{r}}_{2j,2j-1} \qquad (16a)$$

$$\mathbf{r}_{2j+1,2j-1}^T(\ddot{\mathbf{r}}_{2j+1}-\ddot{\mathbf{r}}_{2j-1})=-\dot{\mathbf{r}}_{2j+1,2j-1}^T\dot{\mathbf{r}}_{2j+1,2j-1} \qquad (16b)$$

$$\mathbf{r}_{2j+1,2j}^T(\ddot{\mathbf{r}}_{2j+1}-\ddot{\mathbf{r}}_{2j})=-\dot{\mathbf{r}}_{2j+1,2j}^T\dot{\mathbf{r}}_{2j+1,2j} \qquad (16c)$$

where $\mathbf{G}_{2j-1}$ is the vector sum of the moments of the external forces and force couples acting on the chain starting from body j up till the last body N with respect to the location of particle $2j-1$.

If body "j" is the floating base body in the chain then, two scalar linear momentum equations, similar to Eq. (8), are required to solve for the unknown acceleration components of particle 1. These linear momentum equations equate the sum of the external forces acting on all the bodies in the chain to the time rate of change of the linear momentums of all the equivalent particles that replace the chain and take the form

$$\sum_{i=j}^{N}\mathbf{R}_i=\sum_{i=j}^{N}\sum_{k=2i-1}^{2i}\bar{m}_k\ddot{\mathbf{r}}_k=\bar{m}_{2N+1}\ddot{\mathbf{r}}_{2N+1} \qquad (17)$$

where

$$\bar{m}_h=m_h+\sum_{\substack{k=2i-1\\k\neq h}}^{2i+1}\frac{1}{2}m_{k,h} \qquad (18)$$

In general, for a chain of N bodies, an equivalent system of (2N+1) particles is constructed. By eliminating the coordinates of N particles, we are left with N+1 particles and consequently, 2N+2 unknown acceleration components. To solve for these unknowns, N angular momentum equations can be generated recursively along the chain together with N distance constraints between the pair of particles located on each body. Finally, two linear momentum equations can be used to solve for the unknown acceleration components of particle 1 or for the unknown reaction forces if there is a fixation at point 1. If the chain is closed at its final end, a cut-joint at this end can be used to produce an open chain with the introduction of unknown reaction forces. The cut-joint constraint equations substitute for these unknown reactions.

If bodies "j" and "j−1" are connected by a prismatic joint, then the joint is cut and the original serial chain is separated into two serial chains. The equations of motion are generated recursively along each of the resulting serial chains as discussed above with the added kinematic constraints associated with the prismatic joint. Similar treatment can be used in dealing with all other kinds of lower or higher pair kinematic joints.

In the case of a multi-branch open and/or closed loop system, it can be transformed to a system of serial chains (branchs) by cutting suitable joints. Cut-joint constraints and the associated constraint reaction forces are introduced. For the multi-branch system shown in Fig. 3, the system is divided into 4 chains by cutting the connection joints at points 1, 2, 3 and 4. Equivalent particles are conveniently chosen to locate at the positions of the connection joints and in terms of their Cartesian coordinates the cut-joint constraint equations are easily formulated. These kinematic constraints substitute for the unknown constraint reaction forces that appear explicitly in the linear and angular momentum equations. It is shown also in Fig. 3 that some bodies are connected with the others in many points. In such a case, though the number of particles that dynamically replace the rigid body is three which can be used to define two joints, more particles may be added to describe additional joints.

It should be noted that in this formulation, the kinematic constraints due to some common types of kinematic joints (e.g. revolute or spherical joints) can be automatically eliminated by properly locating the equivalent particles. The remaining kinematic constraints along with the geometric constraints are, in general, either linear or quadratic in the Cartesian coordinates of the particles. Therefore, the coefficients of their Jacobian matrix are constants or linear in the rectangular Cartesian coordinates. Where as in the formulation based on the relative coordinates (Denavit and Hartenberg, 1955 ; Sheth and Uicker, 1972), the constraint equations are derived based on loop closure equations which have the disadvantage that they do not directly determine the positions of the links and points of interest which makes the establishment of the dynamic problem more difficult. Also, the resulting constraint equations are highly nonlinear and contain complex circular functions. The absence of these circular functions in the point coordinate formulation leads to faster convergence and better accuracy. Furthermore, preprocessing the mechanism by the topological graph theory is not necessary as it would be the case with loop constraints.

Also, in comparison with the absolute coordinates formulation, the manual work of the local axes attachment and local coordinates evaluation as well as the use of the rotational variables and the rotation matrices in the absolute coordinate formulation are not required in the point coordinate formulation. This leads to fully computerized analysis and accounts for a reduction in the computational time and memory storage. In addition to that, the constraint equations take much simpler forms as compared with the absolute coordinates.

The elimination of the rotational coordinates, angular velocities and angular accelerations in the presented formulation, leads to possible savings in computation time when this procedure is compared against the absolute or relative coordinate formulation. It has been determined that numerical computations associated with rotational transformation matrices and their corresponding coordinate transformations between reference frames is time consuming and, therefore, if these computations are avoided more efficient codes may be developed (Nikravesh and Attia, 1994). The elimination of rotational coordinates can also be found very beneficial in design sensitivity analysis of multibody systems. In most procedures for design sensitivity analysis, leading to an optimal design process, the derivatives of certain functions with respect to a set of design parameters are required. Analytical evaluation of these derivatives are much simpler if the rotational coordinates are not present and if we only deal with translational coordinates.

Some practical applications of multibody dynamics require one or more bodies in the system to be described as deformable in order to obtain a more realistic dynamic response (Nikravesh and Attia, 1994). Deformable bodies are normally modeled by the finite element technique. Assume that the deformable body is connected to a rigid body described by a set of particles. Then, one or more particles of the rigid body can coincide with one or more nodes of the deformable body in order to describe the kinematic joint between the two bodies. This is a much simpler process that when the rigid body is described by a set of translational and rotational coordinates. In general, the point coordinates have additional advantages over the other systems of coordinates since they are the most suitable coordinates for the graphics routines and the animation programs.

Also, since we are dealing in this formulation with a system of particles instead of rigid bodies, therefore only the laws of particle dynamics are utilized in generating the equations of motion of the mechanical system. This makes the formulation much simpler than the other dynamic formulations which use the rigid body dynamical equations of motion both translational and rotational. In summary, the methodologies presented in this paper have many interesting characteristics which may be found useful in some applications. These methodologies can be combined with other methods to develop even more efficient, accurate, and flexible procedures. It should be noted that there is no single multibody formulation to be considered as the best formulation for general multibody dynamics. Each formulation has its own unique or common features and, therefore, selected features should be adopted to our advantages (Nikravesh and Attia, 1994).

4. Dynamic Analysis of a Multi-Branch Closed-Loop System

The planar three degree-of-freedom platform-type manipulator shown in Fig. 4 is chosen as an example of a multi-closed-loop system. The end-effector platform of the manipulator is constrained by three serial link trains each of which possesses three revolute joints. These three link trains form two independent closed loops. The mechanism is divided into two independent serial branches by cutting the joint at point 5, as

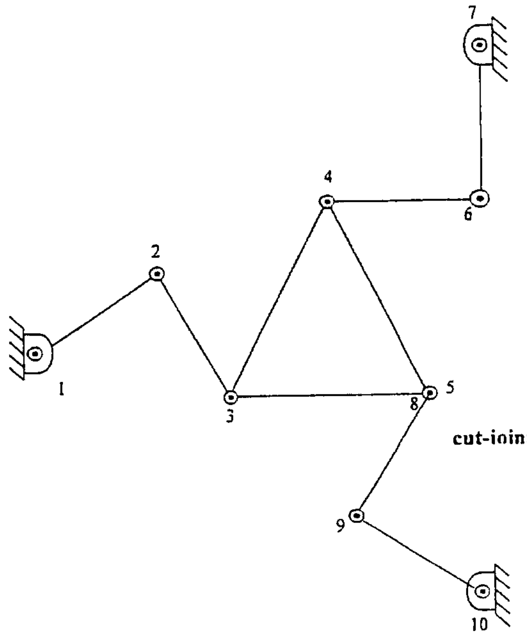

Fig. 4 The manipulator with the equivalent particles

indicated in Fig. 4. Each rigid body is replaced by an equivalent system of 3 particles. Two particles are conveniently located at the centers of the joints connecting the adjacent bodies in the chain, while the Cartesian coordinates of the third particle (not shown in Fig. 4) are expressed in terms of the coordinates of the other two particles with the aid of two distance constraints. As shown in Fig. 4, the platform is a multi-joint body, therefore, additional particle 5 is located at the platform to describe the joint connecting the two separated branches. Locating the particles belonging to adjacent bodies together at the connection joints reduces the total number of particles replacing the whole system and leads to the automatic elimination of the kinematic constraints at these joints. An overall equivalent system of 10 particles is constructed. The equations of motion are generated recursively along each branch as discussed in section 4 while additional cut-joint is needed at the grounded end of the first branch (the location of particle 7). The constraint equations due to cut-joints are introduced in the form,

$$\mathbf{r}_5-\mathbf{r}_8=0,\ \mathbf{r}_7-\mathbf{c}_1=0 \tag{19}$$

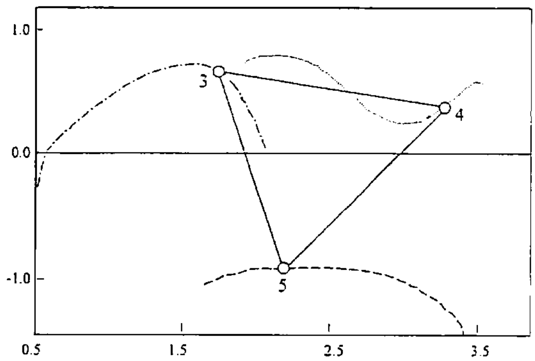

Fig. 5 The trajectory of the platform

where $\mathbf{c}_l$ is a constant known vector. A linear system of 20 algebraic equations can be solved at every time step to determine 16 unknown acceleration components of particles 2, ⋯, 9 as well as 4 the unknown reaction forces at the cut-joints. The motion is started from the rest position under the action of an external force of magnitude 10 N applied at the centre of the end-effector in the x-direction as well as gravity forces. Figure 5 presents the trajectory of the platform in the plane of motion. Verification of the results is done by comparison with the absolute coordinate formulation. It should be noted that for the absolute coordinate formulation, a system of 21+18 differential equations of motion plus algebraic equations of constraints is constructed. Thus a resulting system of 39 differential-algebraic equations should be solved at every time step to determine the unknown accelerations and reaction forces. The reduction in the number of differential equations and in turn the number of integrable variables obtained using the present formulation is considered as an advantage over the absolute coordinate formulation.

5. Conclusions

In the present study, the dynamic modelling of planar mechanisms that consist of a system of rigid bodies is carried out using point coordinates. The system of rigid bodies is replaced by a dynamically equivalent constrained system of particles. Then, the concepts of linear and angular momentums are used to generate the equations of motion. However, they are expressed in terms of the rectangular Cartesian coordinates of a dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces and describing the general motion of the rigid body in terms of a set of Cartesian coordinates without either introducing any rotational coordinates or distributing the external forces and force couples over the particles. The method results in a reduced system of differential-algebraic equations with the absence of the inconvenient rotational coordinates. The methodology is extended to a system of rigid bodies with all common types of kinematic joints, revolute or prismatic. An example of a multi-branch closed-loop system is chosen which indicates the generality and simplicity of the proposed method.

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